# SPRING 2025 MATH 590: QUIZ 12

**Name:**

1. Consider the linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T(x, y) = (2x + 3y, -x + y, 4x + 3y)$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ with $S(x, y, z) = (x - y + z, -x + y - z)$. Letting $\alpha$ denote the standard basis for $\mathbb{R}^2$ and $\beta$ denote the standard basis for $\mathbb{R}^3$, verify the formula $[ST]_\alpha^\alpha = [S]_\beta^\alpha \cdot [T]_\alpha^\beta$. You can use the notation $\alpha = \{e_1, e_2\}$ and $\beta = \{f_1, f_2, f_3\}$. (5 points)

*Solution.* $T(e_1) = T(1, 0) = (2, -1, 4) = 2 \cdot f_1 + -1 \cdot f_2 + 4 \cdot f_3$.

$T(e_2) = T(0, 1) = (3, 1, 3) = 3 \cdot f_1 + 1 \cdot f_2 + 3 \cdot f_3$. Thus, $[T]_\alpha^\beta = \begin{pmatrix} 2 & 3 \\ -1 & 1 \\ 4 & 3 \end{pmatrix}$.

$S(f_1) = S(1, 0, 0) = (1, -1) = 1 \cdot e_1 + -1 \cdot e_2$. $S(f_2) = S(0, 1, 0) = (-1, 1) = -1 \cdot e_1 + 1 \cdot e_2$.

$S(f_3) = S(0, 0, 1) = (1, -1) = 1 \cdot e_1 + -1 \cdot e_2$. Thus, $[S]_\beta^\alpha = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$.

Therefore: $[S]_\beta^\alpha \cdot [T]_\alpha^\beta = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ -1 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ -7 & -5 \end{pmatrix}$.

$ST(e_1) = ST(1, 0) = S(2, -1, 4) = (7, -7) = 7 \cdot e_1 + -7 \cdot e_2$. $ST(e_2) = ST(0, 1) = S(3, 1, 3) = (5, -5)$.

Thus $[ST]_\alpha^\alpha = \begin{pmatrix} 7 & 5 \\ -7 & -5 \end{pmatrix} = [S]_\beta^\alpha \cdot [T]_\alpha^\beta$.

2. Define $T : \mathbb{C}^3 \to \mathbb{C}^3$ by $T(x, y, z) = (x + y, x + y + z, -y + z)$. Find a basis $\alpha \subseteq \mathbb{C}^3$ such that $[T]_\alpha^\alpha$ is in Jordan canonical form. (5 points)

*Solution.* We first note that the matrix of $T$ with respect the standard basis of $\mathbb{R}^3$ is $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Therefore, $p_A(x) = (x - 1)^3$. We now find the change of basis matrix $P$ putting $A$ into its JCF.

$E_1 =$ null space of $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, $E_1$ has basis $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, which is one dimensional. Therefore the JCF $J$ of $A$ is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Now, $(A - 1 \cdot I_3)^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}$. For $v_3$ not in the null space of $(A - 1 \cdot I_3)^2$, we may take $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $v_2 = (A - 1 \cdot I_3) \cdot v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $v_1 = (a - 1 \cdot I_3) \cdot v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Since the columns of $P$ giving the change of basis matrix are $v_1, v_2, v_3$, the basis $\alpha = \{v_1, v_2, v_3\}$, in that order, satisfies $[T]_\alpha^\alpha = J$.